Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00013-1

J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 85–92, 1999

0021-8928/99/$—see front matter

ASYMPTOTIC SPLITTING IN THE THREE-DIMENSIONAL PROBLEM OF LINEAR ELASTICITY FOR ELONGATED BODIES WITH A STRUCTURE†

V. V. YELISEYEV and I. S. ORLOV

St Petersburg

(*Received* 11 *March* 1998)

The equilibrium of linearly-elastic elongated bodies (rods) with an extremely arbitrary geometry and structure subjected to the effects of force and heat is considered. Owing to the presence of a small parameter—the relative thickness—this is a singularly perturbed problem. The asymptotic analysis involves splitting the three-dimensional problem into one- and two-dimensional problems. The one-dimensional problem gives the same result as the classical theory, even when the material is structurally heterogeneous and anisotropic, which invalidates the conventional hypotheses of applied theories. The two-dimensional problems yield not only the parameters of the one-dimensional model, but also a complete solution of the three-dimensional problem. The algorithm used to split the three-dimensional problem is implemented on a computer. It is sometimes more effective than the conventional finite-element, boundary-element and difference methods in the case of elongated bodies.

This paper is a continuation of the previous publications [1, 2]. A different formulation of the asymptotic analysis of the three-dimensional problem for a rod is considered in [3–7].

1. STATEMENT OF THE PROBLEM AND HEURISTIC ARGUMENTS

The geometry of the rods is determined primarily by the axis—a smooth spatial curve on which the radius vector is a function of the arc coordinate $\mathbf{r}(s)$. The section of the rod perpendicular to the axis is geometrically and physically identical for all s. The material is non-homogeneous and orthotropic, and each section lies in a plane of material symmetry. The body is subjected to forces $\mathbf{f}$ distributed arbitrarily over the volume.

The equilibrium of a linearly elastic medium is described by the equations [8]

$$\nabla\cdot\mathbf{T}+\mathbf{f}=0,\quad \operatorname{inc}\mathbf{D}\equiv\nabla\times(\nabla\times\mathbf{D})^T=0$$
$$\mathbf{D}={}^4\mathbf{A}\cdot\cdot\mathbf{T}=\nabla\mathbf{u}^S \tag{1.1}$$

($\mathbf{T}$ and $\mathbf{D}$ are the stress and strain tensors and $\mathbf{u}$ is the displacement vector). The fourth-rank tensor ${}^4\mathbf{A}$ for the given material will be discussed below.

The lateral surface of the rod is assumed to be free: $\mathbf{N}\cdot\mathbf{T}=0$ ($\mathbf{N}$ is the normal to the lateral surface). This does not affect the generality of the argument since the volume forces are arbitrary. The conditions at the ends s = const need special analysis.

The order of smallness of the relative thickness can be expressed by the following representation of the radius vector in the three-dimensional body

$$\mathbf{R}(x_\alpha,s)=\lambda^{-1}\mathbf{r}(s)+\mathbf{x},\quad \mathbf{x}=x_\alpha\mathbf{e}_\alpha,\quad \lambda\to 0$$

where x_α, $\mathbf{e}_\alpha$ are the Cartesian coordinates in the section and the corresponding unit vectors. The coordinate triple $q^1=x_1$, $q^2=x_2$, $q^3=s$ corresponds to the expression of the Hamiltonian operator

$$\nabla=\nabla_\perp+\upsilon^{-1}\mathbf{t}(\partial_s-k_tL) \tag{1.2}$$
$$\nabla_\perp\equiv\mathbf{e}_\alpha\partial/\partial x_\alpha,\quad \mathbf{t}\equiv\partial_s\mathbf{r}\equiv\mathbf{r}'$$
$$\upsilon\equiv\lambda^{-1}+\mathbf{t}\cdot\mathbf{k}_\perp\times\mathbf{x},\quad L\equiv\mathbf{t}\cdot\mathbf{x}\times\nabla_\perp$$

where $\mathbf{k}=\mathbf{k}_\perp+k_t\mathbf{t}$ is the vector of curvature and torsion ($\mathbf{e}_i'=\mathbf{k}\times\mathbf{e}_i$, $\mathbf{e}_3\equiv\mathbf{t}$).

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 93–101, 1999.

The small parameter λ appears in Eqs (1.1) via the ∇-operator so that the asymptotic analysis can begin. A good idea of the method is provided by the linear algebraic system $Cu = f$, $C = C_0 + \lambda C_1$. The case of a regular perturbation ($\det C_0 \neq 0$) is of little interest: $u = u_0 + \lambda u_1 + \ldots$, $C_0 u_0 = f$, $C_0 u_1 = -C_1 u_0, \ldots$, the principal term of the asymptotic form, is found in the first step and we then need to discard formally the small terms in the statement of the problem.

Asymptotic splitting arises if the problem $C_0 u_0$ has a non-trivial solution ($\det C = 0$), when the solution is only partly found in the first step; further steps are required in order to find all the details of the principal term of the asymptotic form from the solvability conditions. This can be done as follows:

$$u = \lambda^{-1} u_0 + u_1 + \ldots,\quad C_0 u_0 = 0 \Rightarrow \psi_n^T (f - C_1 \sum a_k \varphi_k) = 0$$

($\psi_k{:}C_0^T \psi_k = 0$, and a constructive solvability condition of the linear homogeneous problem is used). The initial problem splits into three—with unknowns φ_k, ψ_k and a_k, respectively.

The principal term could contain a different power than λ^{-1}, such as λ^0, λ^{-2}, etc. The number of steps might also be different, but cannot be less than two. For the case of a rod it turned out that the displacement vector $\mathbf{u} = \lambda^{-4}\mathbf{u}_0 + \lambda^{-3}\mathbf{u}_1 + \ldots$. Since the volume forces $\mathbf{f}$ are of the order of unity, it is easy to see that splitting the problem in displacements requires five steps. We will therefore start with the formulation in stresses, for which only three are needed.

2. SPLITTING OF THE STRESS PROBLEM

The stress and strain tensors can be represented in the form

$$\mathbf{T} = \mathbf{T}_\perp + 2\mathbf{S}\mathbf{t}^S + \sigma_t \mathbf{t}\mathbf{t},\quad \mathbf{T}_\perp \equiv T_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta,\quad \mathbf{S} \equiv T_{3\alpha}\mathbf{e}_\alpha$$

$$\mathbf{D} = \mathbf{D}_\perp + \mathbf{d}\mathbf{t}^S + \varepsilon_t \mathbf{t}\mathbf{t},\quad \mathbf{D}_\perp \equiv D_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta,\quad \mathbf{d} \equiv 2D_{3\alpha}\mathbf{e}_\alpha \tag{2.1}$$

The tensor $\mathbf{T}_\perp$ and $\mathbf{D}_\perp$ describe the stress–strain state in the plane of a section, $\mathbf{S}$ and $\mathbf{d}$ are the stress and strain vectors of the transverse shear, σ_t and ε_t are the extension stress and strain. Note that it is usual in applications to drop the stresses $\mathbf{T}_\perp$ as secondary. But it is not possible to do this in the case of rods with structural heterogeneity, as all the components of $\mathbf{T}$ may be of the same order. Owing to the anisotropy, even small components of $\mathbf{T}$ must be calculated if they operate on small areas which cohere only slightly.

Substituting $\mathbf{T}$ from (2.1) into the equation of the balance of forces (1.1) and using (1.2), we arrive at the system

$$\nabla_\perp \cdot (\upsilon\mathbf{T}_\perp) + \dot{\mathbf{S}} - k_t(L\mathbf{S} + \mathbf{S}\times\mathbf{t}) - \sigma_t \mathbf{t}\times\mathbf{k}_\perp + \upsilon\mathbf{f}_\perp = 0 \tag{2.2}$$
$$\nabla_\perp \cdot (\upsilon\mathbf{S}) + \dot{\sigma}_t - k_t L\sigma_t + \mathbf{t}\cdot\mathbf{k}_\perp \times \mathbf{S} + \upsilon f_t = 0$$

We have introduced differentiation in a moving basis: for the scalar $\dot{\sigma}_t \equiv \partial_s \sigma_t$, for the vector $\dot{\mathbf{S}} \equiv \dot{S}_i \mathbf{e}_i = \partial_s \mathbf{S} - \mathbf{k}\times\mathbf{S}$, etc.

The boundary condition $\mathbf{N}\cdot\mathbf{T} = 0$ on the free lateral surface can be represented in the form

$$\mathbf{n}\cdot(\upsilon\mathbf{T}_\perp)|_{\partial F} = k_t \mathbf{x}\times\mathbf{n}\cdot\mathbf{t}\mathbf{S},\quad \mathbf{n}\cdot(\upsilon\mathbf{S}|_{\partial F} = k_t \mathbf{x}\times\mathbf{n}\cdot\mathbf{t}\sigma_t \tag{2.3}$$

where $\mathbf{n}$ is the unit normal to the contour ∂F of the section in its plane. Relations (2.3) can be derived by specifying the equation of the lateral surface $\varphi(x_\alpha) = 0$ and calculating $\mathbf{N} = \nabla\varphi/|\nabla\varphi|$ with representation (1.2) for ∇.

The statement of the problem in stresses also contains the equation of compatibility of the strains (1.1). From (1.2) and (2.1), we obtain

$$\Delta_\perp \varepsilon_\perp = \nabla_\perp \cdot \nabla_\perp \cdot \mathbf{D}_\perp + \ldots,\quad (\varepsilon_\perp \equiv \operatorname{tr}\mathbf{D}_\perp)$$

$$\Delta_\perp \mathbf{d} = \nabla_\perp \nabla_\perp \cdot \mathbf{d} + \ldots,\quad \nabla_\perp \nabla_\perp \varepsilon_t = \mathbf{E}_\perp \Delta_\perp \varepsilon_t + \ldots\ (\mathbf{E}_\perp \equiv \mathbf{e}_\alpha \mathbf{e}_\alpha) \tag{2.4}$$

in which only those terms which are important as $\lambda \to 0$ are included.

For the given orthotropic material Hooke's law takes the form

$$\mathbf{D}_\perp = {}^4\mathbf{A}_\perp \cdot\cdot \mathbf{T}_\perp + \mathbf{A}_1 \sigma_t,\quad \mathbf{d} = \mathbf{A}_2 \cdot \mathbf{S},\quad \varepsilon_t = \mathbf{A}_1 \cdot\cdot \mathbf{T}_\perp + \frac{1}{E_t}\sigma_t \tag{2.5}$$

($^4\mathbf{A}_\perp$, $\mathbf{A}_1$, $\mathbf{A}_2$ are tensors in the plane of the section, of fourth and second rank, respectively).

In the general case the conditions at the ends s = const require the use of the method of matched asymptotic expansions [9].

We will seek a solution of problem (2.2)–(2.5) in the form of a series in integer powers of λ

$$\mathbf{T} = \lambda^m(\mathbf{T}^{(0)} + \lambda\mathbf{T}^{(1)} + ...)$$

The values $m = 0$ and $m = -1$ lead to contradictions. The only other possibility is $m = -2$. For the principal term $\mathbf{T}^{(0)}$ we will then have

$$\nabla_\perp \cdot \mathbf{T}_\perp^{(0)} = 0, \ \mathbf{n} \cdot \mathbf{T}_\perp^{(0)} |_{\partial F} = 0 \tag{2.6}$$

$$\Delta_\perp \varepsilon_\perp^{(0)} = \nabla_\perp \cdot \nabla_\perp \cdot \mathbf{D}_\perp^{(0)}, \ \nabla_\perp \nabla_\perp \varepsilon_t^{(0)} = 0$$

$$\nabla_\perp \cdot \mathbf{S}^{(0)} = 0, \ \mathbf{n} \cdot \mathbf{S}^{(0)} |_{\partial F} = 0, \ \Delta_\perp \mathbf{d}^{(0)} = \nabla_\perp \nabla_\perp \cdot \mathbf{d}^{(0)} \tag{2.7}$$

We will start our solution of problem (2.6) with the last equation

$$\varepsilon_t^{(0)} = A(s) + \mathbf{B}(s) \cdot \mathbf{x} \tag{2.8}$$

Even in a non-homogeneous and anisotropic body the deformation ε_t is linearly distributed over the section (as $\lambda \to 0$). It then follows from (2.5) that

$$\mathbf{D}_\perp = (^4\mathbf{A}_\perp - E_t \mathbf{A}_1 \mathbf{A}_1) \cdot\cdot \mathbf{T}_\perp + E_t \mathbf{A}_1 (A + \mathbf{B} \cdot \mathbf{x}) \tag{2.9}$$

But this is similar to Hooke's law in the plane problem with initial strains. The equations and boundary conditions (2.6) totally define a solution of the form

$$\mathbf{T}_\perp^{(0)} = A\mathbf{T}_A + \mathbf{B} \cdot \mathbf{e}_\alpha \mathbf{T}_{B\alpha} \tag{2.10}$$

Here $\mathbf{T}_A$ and $\mathbf{T}_{B\alpha}$ are solutions of the three problems in the plane of the section with initial strains $\mathbf{D}_\perp^0 = E_t A_1$ and $E_t A_1 x_\alpha$, respectively. These strains are incompatible in the case of a non-homogeneous material, and so $\mathbf{T}_\perp^0 \neq 0$.

In addition to these plane problems, we need to solve problem (2.7) for the tangential stress vector $\mathbf{S}^{(0)}$. Putting $\nabla_\perp \times \mathbf{d}^{(0)} \cdot \mathbf{t} = 2C$, we obtain $C = C(s)$ from the last equation of (2.7). Then from (2.7) and (2.5) we arrive at the following relation (the section is assumed to be simply-connected)

$$\mathbf{S}^{(0)} = C\nabla_\perp g \times \mathbf{t} \tag{2.11}$$

$$g : \nabla_\perp \cdot [-(\mathbf{t} \times \mathbf{A}_2 \times \mathbf{t}) \cdot \nabla_\perp g] = -2, \ g|_{\partial F} = 0 \tag{2.12}$$

The last component of $\mathbf{T}^0$ is

$$\sigma_t^{(0)} = A\sigma_A + B_\alpha \sigma_\alpha \ \ (B_\alpha \equiv \mathbf{B} \cdot \mathbf{e}_\alpha)$$

$$\sigma_A \equiv E_t(1 - \mathbf{A}_1 \cdot\cdot \mathbf{T}_A), \ \sigma_\alpha \equiv E_t(x_\alpha - \mathbf{A}_1 \cdot\cdot \mathbf{T}_{B\alpha}) \tag{2.13}$$

The two-dimensional problems have thus been stated completely (for $\mathbf{T}_A$, $\mathbf{T}_{B\alpha}$ and g). It is now necessary to determine the functions of the arc coordinate A, $\mathbf{B}$ and C, by picking out the one-dimensional problem. This is found from the corresponding solvability conditions. The two-dimensional problems

$$\nabla_\perp \cdot \mathbf{T}_\perp + \mathbf{f}_\perp = 0, \ \mathbf{n} \cdot \mathbf{T}_\perp |_{\partial F} = \mathbf{p}_\perp \tag{2.14}$$

$$\nabla_\perp \cdot \mathbf{S} + f = 0, \ \mathbf{n} \cdot \mathbf{S} |_{\partial F} = p$$

are solvable only under the conditions (everywhere in this and the next two sections integration is over the volume F and surface ∂F)

$$\int \mathbf{f}_\perp dF + \oint \mathbf{p}_\perp dl = 0$$

$$\int \mathbf{x} \times \mathbf{f}_\perp \cdot \mathbf{t} dF + \oint \mathbf{x} \times \mathbf{p}_\perp \cdot \mathbf{t} dl = 0 \tag{2.15}$$

$$\int f\, dF + \oint p dl = 0, \ \int (f\mathbf{x} - \mathbf{S}) dF + \oint p\mathbf{x} dl = 0$$

(the last relation is an auxiliary identity). Relations (2.2) and (2.3) have the same form as (2.14); for them, Eqs (2.15) reduce to the following well-known equations of the balance of forces and moments for a rod

$$\mathbf{Q}' + \mathbf{q} = 0, \ \ \mathbf{M}' + \lambda^{-1}\mathbf{t} \times \mathbf{Q} + \mathbf{m} = 0 \tag{2.16}$$

$$\mathbf{Q} \equiv \int (\mathbf{S} + \sigma_t \mathbf{t}) dF, \ \ \mathbf{M} \equiv \int \mathbf{x} \times (\mathbf{S} + \sigma_t \mathbf{t}) dF \tag{2.17}$$

$$\mathbf{q} \equiv \int \mathbf{f} \upsilon \, dF, \ \ \mathbf{m} \equiv \int \mathbf{x} \times \mathbf{f} \upsilon \, dF$$

The vectors **Q** and **M**, like **T**, can be represented by power series in λ. For the principal terms of (2.16) and (2.17), we must have $\mathbf{Q}^{(0)\prime} = 0$, $\mathbf{t} \times \mathbf{Q}^{(0)} = 0 \Rightarrow \mathbf{Q}^{(0)} = 0$ for a rod which is not straight. Then we can eliminate $A(s)$

$$0 = Q_t^{(0)} = \int \sigma_t^{(0)} dF \Rightarrow$$

$$A = -B_\alpha \int \sigma_a dF / \int \sigma_A dF \tag{2.18}$$

3. THE DISPLACEMENT FIELD. THE ELASTICITY RELATIONS

We find $\mathbf{u}(\mathbf{R})$ by integrating the equation $\nabla \mathbf{u}^S = \mathbf{D} = {}^4\mathbf{A} \cdot\cdot \mathbf{T}$. We have

$$\mathbf{D}_\perp = (\nabla_\perp \mathbf{u}_\perp)^S, \ \ \varepsilon_t = \upsilon^{-1}[\dot{u}_t - k_t L u_t + \mathbf{k}_\perp \times \mathbf{u}_\perp \cdot \mathbf{t}]$$

$$\mathbf{d} = \nabla_\perp u_t + \upsilon^{-1}[\dot{\mathbf{u}}_\perp - k_t L \mathbf{u}_\perp + k_t \mathbf{t} \times \mathbf{u}_\perp + \mathbf{k}_\perp \times u_t \mathbf{t}] \tag{3.1}$$

$$(\mathbf{u} = \mathbf{u}_\perp + u_t \mathbf{t})$$

The expansion for **u** must contain λ^{-2}, the stresses. But the only expansion which does not involve a contradiction has the form

$$\mathbf{u} = \lambda^{-4}(\mathbf{u}^{(0)} + \lambda \mathbf{u}^{(1)} + \ldots)$$

In the first step we obtain

$$\mathbf{u}_\perp^{(0)} = \mathbf{U}_\perp^{(0)}(s) + \vartheta_t^{(0)}(s) \mathbf{t} \times \mathbf{x}, \ \ u_t^{(0)} = U_t^{(0)}(s)$$

In the second

$$\mathbf{u}_\perp^{(1)} = \mathbf{U}_\perp^{(1)}(s) + \vartheta_t^{(1)}(s) \mathbf{t} \times \mathbf{x}, \ \ \dot{U}_t^{(0)} + \mathbf{k}_\perp \times \mathbf{U}_\perp^{(0)} \cdot \mathbf{t} = 0$$

$$\vartheta_t^{(0)} = 0, \ \ \dot{\mathbf{U}}_\perp^{(0)} + k_t \mathbf{t} \times \mathbf{U}_\perp^{(0)} + \mathbf{k}_\perp \times \mathbf{U}_t^{(0)} \mathbf{t} \equiv \boldsymbol{\vartheta}_\perp^{(1)}(s) \times \mathbf{t}$$

The last three equations merely show that

$$\mathbf{u}^{(0)} = \mathbf{U}^{(0)}(s), \ \ \mathbf{U}^{(0)\prime} = \boldsymbol{\vartheta}^{(1)} \times \mathbf{t} \tag{3.2}$$

This is fully consistent with Kirchhoff's classical theory of rods in its linear form.

The stresses appear only at the third step. Omitting the simple algebra, the final result for the elasticity relations of the one-dimensional problem is

$$\mathbf{M}_\perp^{(0)} = \mathbf{a}_\perp \cdot \boldsymbol{\vartheta}^{(1)\prime}$$

$$\mathbf{a}_\perp \equiv -\mathbf{t} \times \int \mathbf{x} \left(\sigma_\alpha - \sigma_A \frac{\int \sigma_\alpha dF}{\int \sigma_A dF} \right) dF \mathbf{e}_\alpha \times \mathbf{t} \tag{3.3}$$

$$M_t^{(0)} = a_t \vartheta^{(1)\prime} \cdot \mathbf{t}, \ \ a_t \equiv 2 \int g dF$$

The second-rank tensor $\mathbf{a}_\perp$ is the flexural rigidity and the scalar a_t is the torsional rigidity.

Another outcome of the third step is the equation

$$\boldsymbol{\vartheta}^{(1)\prime} = \mathbf{B} \times \mathbf{t} + C\mathbf{t} \tag{3.4}$$

which associates the one-dimensional and two-dimensional problems. Finding $\mathbf{B}(s)$ and $C(s)$ from the one-dimensional problem, we can determine the three-dimensional stress field from the formulae of Section 2. There will be a second reference to the two-dimensional solutions in the section, which are first needed to calculate the rigidities $\mathbf{a}_\perp$, d_t.

4. TEMPERATURE DEFORMATIONS

The splitting algorithm is easily applied to temperature deformations. Equations (2.5) are replaced by the relations

$$\mathbf{D}_\perp = {}^4\mathbf{A}_\perp \cdot\cdot\, \mathbf{T}_\perp + \mathbf{A}_1\sigma_t + \boldsymbol{\alpha}_\perp\Theta, \quad \mathbf{d} = \mathbf{A}_2 \cdot \mathbf{S}$$
$$\varepsilon_t = \mathbf{A}_1 \cdot\cdot\, \mathbf{T}_\perp + \frac{1}{E_t}\sigma_t + \alpha_t\Theta \tag{4.1}$$

where $\boldsymbol{\alpha}_\perp + \alpha_t\mathbf{tt}$ is the tensor of the coefficients of thermal expansion and Θ is the change in temperature.

The series for the stress tensor does not need to include negative powers: $\mathbf{T} = \mathbf{T}^{(0)} + \lambda\mathbf{T}^{(1)} + \ldots$ (if $\mathbf{f} = 0$). Since (2.9) will then be replaced by

$$\mathbf{D}_\perp = \left({}^4\mathbf{A}_\perp - E_t\mathbf{A}_1\mathbf{A}_1\right)\cdot\cdot\,\mathbf{T}_\perp + E_t\mathbf{A}_1(A + \mathbf{B}\cdot\mathbf{x}) + \left\{\left(\boldsymbol{\alpha}_\perp - E_t\alpha_t\mathbf{A}_1\right)\Theta\right\} \tag{4.2}$$

the three problems for $\mathbf{T}_A$ and $\mathbf{T}_{B\alpha}$ are now joined by a fourth—for $\mathbf{T}_\Theta$ with initial deformation in the form of the term in braces.

The only change in the one-dimensional problem is the expression for the bending moment

$$\mathbf{M}_\perp^{(0)} = \mathbf{a}_\perp \cdot \boldsymbol{\vartheta}^{(1)\prime} + \mathbf{M}_\Theta \tag{4.3}$$
$$\mathbf{M}_\Theta \equiv \int \mathbf{x}\left(\sigma_\Theta - \sigma_A \frac{\int \sigma_\Theta dF}{\int \sigma_A dF}\right) dF \times \mathbf{t}$$
$$\sigma_\Theta \equiv -E_t\left(\mathbf{A}_1 \cdot\cdot\, \mathbf{T}_\Theta + \alpha_t\Theta\right)$$

5. SOLUTION OF THE ONE-DIMENSIONAL PROBLEM

When the geometry and rigidity $\mathbf{a}_\perp$ of the rod are arbitrary, the one-dimensional problem can only be solved numerically (it is obviously necessary to use a computer for the two-dimensional problems). But splitting led to a one-dimensional model with an inextensible axis, with computational problems arising if the rod was not "curved enough". We shall therefore use the following one-dimensional model with extension

$$\mathbf{Q}' + \mathbf{q} = 0, \quad \mathbf{M}' + \mathbf{t}\times\mathbf{Q} + \mathbf{m} = 0, \quad \mathbf{u}' - \boldsymbol{\vartheta}\times\mathbf{t} = \gamma\mathbf{t} \tag{5.1}$$
$$\mathbf{M} = \mathbf{a}\cdot\boldsymbol{\vartheta}' + \mathbf{M}_\Theta, \quad \mathbf{Q}\cdot\mathbf{t} = b\gamma, \quad \mathbf{a} \equiv \mathbf{a}_\perp + a_t\mathbf{tt}$$

where $b > 0$ is the tensile stiffness (this is a regularizing parameter whose exact value is not as important as the definition of $\mathbf{a}$).

Equations (5.1) with $\mathbf{u}$ and $\boldsymbol{\vartheta}$ given at the ends allow of a variational formulation: it is required to minimize the Lagrange functional

$$J(\mathbf{u},\boldsymbol{\vartheta}) = \int_0^l \left[\frac{1}{2}\left(\boldsymbol{\vartheta}'\cdot\mathbf{a}\cdot\boldsymbol{\vartheta}' + b\gamma_t^2\right) + \mathbf{M}_\Theta\cdot\boldsymbol{\vartheta}' - \mathbf{q}\cdot\mathbf{u} - \mathbf{m}\cdot\boldsymbol{\vartheta}\right] ds \tag{5.2}$$

under the additional constraint $\boldsymbol{\vartheta}_\perp = \mathbf{t}\times\mathbf{u}'$. Approximating the solution by a linear combination with a finite number of variable coefficients, we construct a computational algorithm of the finite-element method. The most effective one is with an approximation in the form of the exact solution of Eqs (5.1) with given values of $\mathbf{u}$ and $\boldsymbol{\vartheta}$ at the ends of the interval (finite element) and no distributed effects $\mathbf{q}$, $\mathbf{m}$ and $\mathbf{M}_\Theta$. If the effect is concentrated at nodes (element ends), we obtain the exact solution of the problem as a whole.

We will find the stiffness matrix of a finite element. We have

$$\mathbf{Q} = \text{const}, \quad \mathbf{M} + \mathbf{r} \times \mathbf{Q} = \mathbf{M}_0 = \text{const} \tag{5.3}$$

$$\boldsymbol{\vartheta}\big|_0^l \equiv \mathbf{h} = \mathbf{S}_{hM}(l) \cdot \mathbf{M}_0 + \mathbf{S}_{hQ}(l) \cdot \mathbf{Q}$$

$$(\mathbf{u} - \boldsymbol{\vartheta} \times \mathbf{r})\big|_0^l \equiv \mathbf{v} = \mathbf{S}_{hQ}^T(l) \cdot \mathbf{M}_0 + \mathbf{S}_{vQ}(l) \cdot \mathbf{Q} \tag{5.4}$$

$$\mathbf{S}_{hM}(x) \equiv \int_0^x \mathbf{a}_- ds, \quad \mathbf{S}_{hQ}(x) \equiv -\int_0^x \mathbf{a}^{-1} \times \mathbf{r} ds$$
$$\mathbf{S}_{vQ}(x) \equiv \int_0^x \left[b^{-1}\mathbf{tt} - \mathbf{r} \times \mathbf{a}^{-1} \times \mathbf{r} \right] ds \tag{5.5}$$

From (5.4) we obtain

$$\mathbf{M}_0 = \mathbf{C}_{Mh} \cdot \mathbf{h} + \mathbf{C}_{Mv} \cdot \mathbf{v}, \quad \mathbf{Q} = \mathbf{C}_{Mv}^T \cdot \mathbf{h} + \mathbf{C}_{Qv} \cdot \mathbf{v}$$
$$\mathbf{C}_{Qv} \equiv \left(\mathbf{S}_{vQ}(l) - \mathbf{S}_{hQ}^T(l) \cdot \mathbf{S}_{hM}^{-1}(l) \cdot \mathbf{S}_{hQ}(l) \right)^{-1}$$
$$\mathbf{C}_{Mv} \equiv -\mathbf{S}_{hM}^{-1}(l) \cdot \mathbf{S}_{hQ}(l) \cdot \mathbf{C}_{Qv} \tag{5.6}$$
$$\mathbf{C}_{Mh} \equiv \mathbf{S}_{hM}^{-1}(l) \cdot \left(\mathbf{E} - \mathbf{S}_{hQ}(l) \cdot \mathbf{C}_{Mv}^T \right)$$

Substituting $\mathbf{M}_0$ and $\mathbf{Q}$ from (5.6) into the expression for the energy of the rod

$$J = \frac{1}{2} \int_0^l \left[\mathbf{M} \cdot \boldsymbol{\vartheta}' + \mathbf{Q} \cdot (\mathbf{u}' - \boldsymbol{\vartheta} \times \mathbf{t}) \right] ds = \mathbf{M}_0 \cdot \mathbf{h} + \mathbf{Q} \cdot \mathbf{v}$$

we obtain J in the form of a quadratic form of $\mathbf{h}$ and $\mathbf{v}$. The coefficients of the stiffness matrix of an element can be found by changing to nodal variables $\mathbf{u}(0)$, $\boldsymbol{\vartheta}(0)$, $\mathbf{u}(l)$, $\boldsymbol{\vartheta}(l)$.

The contribution made by distributed loads to the right-hand side of the system of equations of the finite-element method is given by the linear terms in the Lagrange functional (5.2). The approximations $\mathbf{u}(s)$, $\boldsymbol{\vartheta}(s)$, $\boldsymbol{\vartheta}'(s)$ can be computed from the formulae

$$\mathbf{u} = \mathbf{u}(0) + \boldsymbol{\vartheta}(0) \times (\mathbf{r} - \mathbf{r}(0)) + \left(\mathbf{S}_{hQ}^T - \mathbf{r} \times \mathbf{S}_{hM} \right) \cdot \mathbf{M}_0 + \left(\mathbf{S}_{vQ} - \mathbf{r} \times \mathbf{S}_{hQ} \right) \cdot \mathbf{Q}$$
$$\boldsymbol{\vartheta} = \boldsymbol{\vartheta}(0) + \mathbf{S}_{hM} \cdot \mathbf{M}_0 + \mathbf{S}_{hQ} \cdot \mathbf{Q}$$
$$\boldsymbol{\vartheta}' = \mathbf{a}^{-1} \cdot (\mathbf{M}_0 - \mathbf{t} \times \mathbf{Q})$$

The vectors $\mathbf{M}_0$ and $\mathbf{Q}$ can be expressed in terms of nodal displacements and rotations as in (5.6).

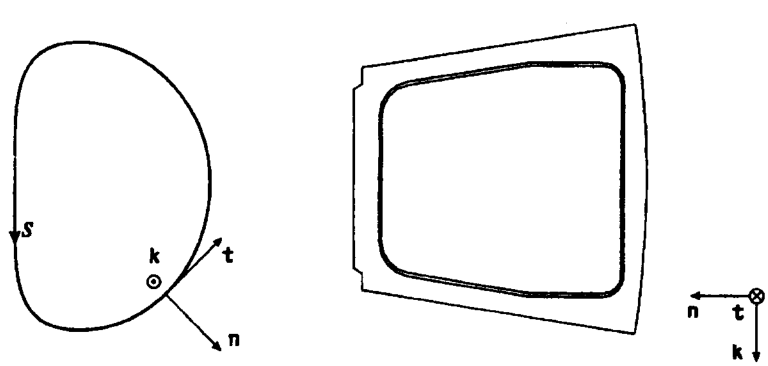

Fig. 1.

6. EXAMPLE: A COOLED SOLENOID

The above algorithm for the three-dimensional problem was used to calculate the stress of a thin closed magnetic coil of a thermonuclear reactor. The solenoid is cooled to a superconducting state. The axis of the coil is a D-shaped plane curve, and the section is a trapezium (Fig. 1). The coil consists of a winding which is placed inside a steel band and separated from it by a thin insulating layer. The winding is made of a composite of periodic structure (whose properties were calculated by the averaging method [10]).

We will first consider the one-dimensional problem (5.1) for a rod with a closed axis. We have $\mathbf{q} = 0$, $\mathbf{m} = 0$, so that $\mathbf{Q}$ = const, $\mathbf{M} + \mathbf{r} \times \mathbf{Q} = \mathbf{M}_0$ = const. Integrating the elasticity relations, we obtain expressions for the displacement and rotation vectors. If there are no displacements, rotations, forces or moments in the uncooled rod, the following conditions for closure of the axis are obtained from these expressions

$$\oint \mathbf{a}^{-1} \cdot [\mathbf{M}_0 - \mathbf{M}_\Theta - \mathbf{r} \times \mathbf{Q}] ds = 0$$

$$\oint \left\{ \mathbf{r} \times \mathbf{a}^{-1} \cdot [\mathbf{M}_0 - \mathbf{M}_\Theta - \mathbf{r} \times \mathbf{Q}] + b^{-1} \mathbf{Q} \cdot \mathbf{tt} \right\} ds = 0 \tag{6.1}$$

In the special case considered here the rod axis is a plane curve, the rod is untwisted and its elastic properties and the temperature field are independent of the arc coordinate s. The tensor $\mathbf{a}^{-1}$ and temperature moment $\mathbf{M}_\theta$ can be represented as follows (cf. (3.3) and (4.3))

$$\mathbf{a}^{-1} = A_1 \mathbf{nn} + A_2 \mathbf{kk} + A_{12}(\mathbf{kn} + \mathbf{nk}) + A_t \mathbf{tt}$$

$$\mathbf{M}_\Theta = M_1 \mathbf{n} + M_2 \mathbf{k}$$

Here $A_1, A_2, A_{12}, A_t, M_1, M_2$ are constants, $\mathbf{n} = \mathbf{e}_1$ is the principal normal to the axis and $\mathbf{k} = \mathbf{e}_2 \equiv \mathbf{n} \times \mathbf{t}$ is the unit normal to the plane of the axis. In this problem the closure conditions (6.1) are satisfied by the following values of the force $\mathbf{Q}$ and moment $\mathbf{M}$

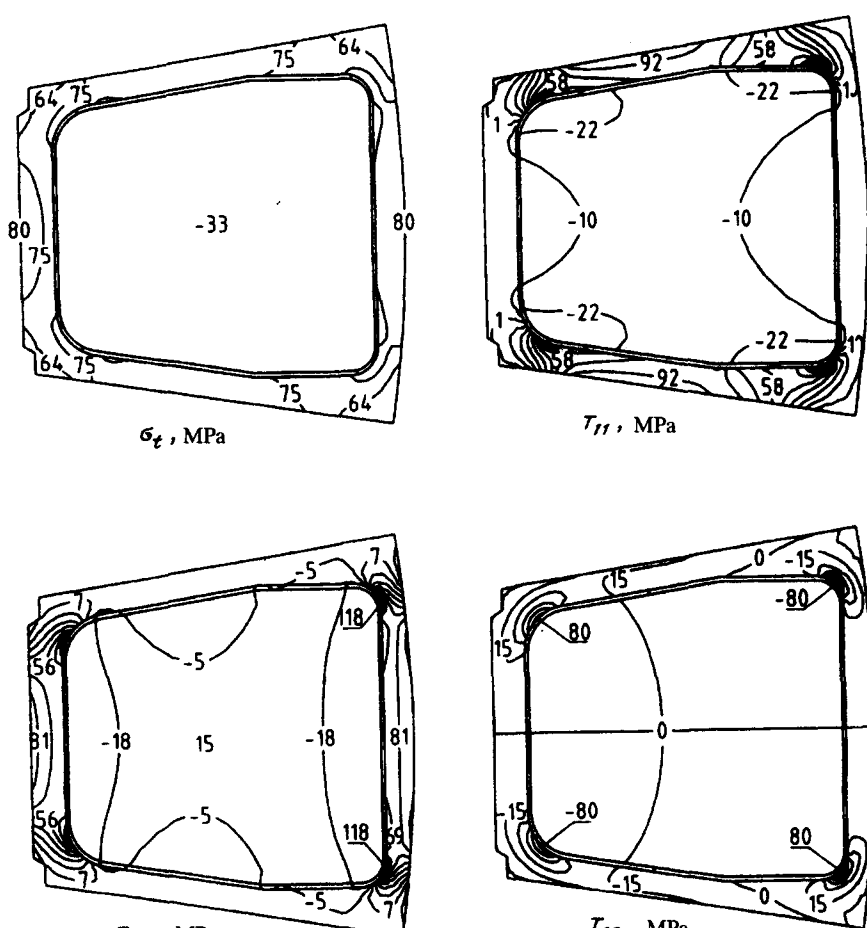

Fig. 2.

Table 1

No.	E_1 10^9 Pa	E_2 10^9 Pa	E_t 10^9 Pa	G_{12} 10^9 Pa	G_{23} 10^9 Pa	G_{31} 10^9 Pa	ν_{12}	ν_{23}	ν_{31}
1	208	208	208	83.9	83.9	83.9	0.24	0.24	0.24
2	20.0	20.0	12.0	6.00	7.52	8.50	0.33	0.33	0.17
3	86.1	81.7	124	23.3	34.0	37.1	0.20	0.17	0.26

$$\mathbf{Q}=0, \quad \mathbf{M}=\left(M_2+\frac{A_{12}}{A_2}M_1\right)\mathbf{k}$$

The stiffnesses of the rod for the one-dimensional model are found by solving the problems on a section. These are then used to calculate the stress tensor over the volume of the rod from the value of the moment $\mathbf{M}$.

We considered a section consisting of three orthotropic materials where $\alpha_1 = \alpha_2 = \alpha_t = 1.1 \times 10^{-5}$ K^{-1} for the casing and $\alpha_1 = \alpha_2 = \alpha_t = 9.3 \times 10^{-6}$ K^{-1} for the insulation and winding; other properties of the materials are given in Table 1. The dimensions of the section are of the order of one metre. The problems on the section were solved by the finite-element method (using eight-node quadratic isoparametric elements). We found the stiffnesses $\mathbf{a}$ (formulae (3.3)), $b \equiv \int_F E_t dF$, the value of the temperature moment $\mathbf{M}_\theta$ (4.3), solved the one-dimensional problem and calculated the constants $\mathbf{B}$ (3.4) and A (2.18).

We thereby obtained the stress tensor field on section (Fig. 2). It is worth noting that when the coil is cooled all four non-zero components are of the same order: this is untypical of traditional models of a rod.

REFERENCES

1. YELISEYEV, V. V., One- and three-dimensional models of elastic rods. *Tr. St Petersburg. Gos. Tekh. Univ.*, 1993, 446, 29–52.
2. YELISEYEV, V. V., *The Mechanics of Elastic Rods*. St Petersburg Gos. Tekh. Univ., St Petersburg, 1994.
3. PONYATOVSKII, V. V., The use of an asymptotic method of integration in the problem of the equilibrium of a thin beam, loaded arbitrarily on the lateral surface. *Inzh. Zh. MTT*, 1968, 5, 139–143.
4. GALAKTIONOV, Ye. V. and TROPP, E. A., An asymptotic method of calculating thermoelastic stresses in a slender rod. *Izv. Akad. Nauk SSSR. Ser. Fiz.*, 1976, **40**, 1399–1406.
5. BERDICHEVSKII, V. L. and STAROSEL'SKII, L. A., The theory of curvilinear Timoshenko-type rods. *Prikl. Mat. Mekh.*, 1983, **47**, 1015–1024.
6. BUTUZOV, V. F. and URAZGIL'DINA, T. A., The asymptotic solution of a quasistatic thermoelasticity problem for a slender rod. *Prikl. Mat. Mekh.*, 1987, **51**, 989–999.
7. ILYUKHIN, A. A., Derivation of the relations of the theory of elastic rods. *The Mechanics of Solids*. 1990, 22, 81–92. Naukova Dumka, Kiev.
8. LUR'YE, A. I., *The Theory of Elasticity*. Nauka, Moscow, 1970.
9. NAYFEH, A. H., *Perturbation Methods*. John Wiley, New York, 1972.
10. BAKHVALOV, N. S. and PANASENKO, G. P., *Averaging Processes in Periodic Media*. Nauka, Moscow, 1984.

Translated by R.L.